

Spectrum of the Product of Independent Random Gaussian Matrices

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We show that the eigenvalue density of a product $X = X_1 X_2 \cdots X_M$ of M independent $N \times N$ Gaussian random matrices in the limit $N \rightarrow \infty$ is rotationally symmetric in the complex plane and is given by a simple expression $\rho(z, \bar{z}) = \frac{1}{M\pi} \sigma^{-\frac{2}{M}} |z|^{-2+\frac{2}{M}}$ for $|z| \leq \sigma$, and is zero for $|z| > \sigma$. The parameter σ corresponds to the radius of the circular support and is related to the amplitude of the Gaussian fluctuations. This form of the eigenvalue density is highly universal. It is identical for products of Gaussian Hermitian, non-Hermitian, real or complex random matrices. It does not change even if the matrices in the product are taken from different Gaussian ensembles. We present a self-contained derivation of this result using a planar diagrammatic technique. Additionally, we conjecture that this distribution also holds for any matrices whose elements are independent, centered random variables with a finite variance or even more generally for matrices which fulfill Pastur-Lindeberg's condition. We provide a numerical evidence supporting this conjecture.

I. INTRODUCTION

Initiated by Wigner more than 50 years ago and developed by Dyson, Mehta and others, Random Matrix Theory (RMT) has been successfully applied to various problems ranging from fundamental physics (for a comprehensive review see [1]) to engineering and financial applications [2]. One of the reasons of such a wide applicability is the universality of many results predicted by RMT. Let us take as an example the problem addressed by Wigner, that is how to determine the energy spectrum and level spacing distribution of a many-body quantum system. Due to many degrees of freedom and sophisticated nature of interactions one has to turn to a statistical description. However, in contrast to statistical mechanics where one fixes the Hamiltonian and averages over possible states of the system, Wigner proposed to treat the very Hamiltonian as a random operator, which in turn can be represented as a large random matrix. Relevant properties of such a matrix are determined by symmetries of the problem. The great discovery of RMT is that many observables are the same for various statistical ensembles of random matrices.

To illustrate this, let us cite two classical results of RMT. The eigenvalue density of a real symmetric or complex Hermitian $N \times N$ matrix, whose entries in the upper/lower triangle are independent, identically distributed random variables with a finite variance equal to σ^2/N , converges for $N \rightarrow \infty$ to a limiting distribution

$$\rho(\lambda) = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - \lambda^2}, \quad \text{for } \lambda \in [-2\sigma, 2\sigma], \quad (1)$$

known as Wigner's semicircle distribution, one of the best known results of classical RMT. The class of matrices whose spectrum converges to the limit law (1) is actually much broader and embraces matrices with entries being independent random variables which fulfill Pastur-Lindeberg's condition [3]. This is an example of macroscopic universality of random matrices. In this paper we concentrate on macroscopic properties and do not discuss microscopic properties of eigenvalue statistics.

An analogous formula for a non-Hermitian random matrix, which is another example of a macroscopic law, reads

$$\rho(z, \bar{z}) = \begin{cases} \frac{1}{\pi\sigma^2} & \text{for } |z| \leq \sigma, \\ 0 & \text{for } |z| > \sigma, \end{cases} \quad (2)$$

where $z = x + iy$ is a complex number. The distribution (2) is called Girko-Ginibre's distribution. The eigenvalue density has a rotational symmetry in the complex plane and is uniform inside the circle of radius σ . More generally, if a matrix has independent but not identically distributed Hermitian and anti-Hermitian degrees of freedom [4], the

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limit law (2) assumes an elliptic form

$$\rho(z, \bar{z}) = \begin{cases} \frac{1}{(1-\tau^2)\pi\sigma^2} & \text{for } \frac{x^2}{\sigma^2(1+\tau)^2} + \frac{y^2}{\sigma^2(1-\tau)^2} \leq 1, \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

where $\sigma^2 > 0$ is an effective scale parameter and $\tau \in [-1, 1]$ is a flatness of the ellipse. For $\tau = 0$ one recovers the circular law (2). For $\tau \rightarrow \pm 1$ the support of the distribution (3) reduces to a cut $[-2\sigma, 2\sigma]$ on the real (for $\tau \rightarrow 1$) or imaginary (for $\tau \rightarrow -1$) axis and the distribution itself reduces to a Wigner law (1), as one can see by projecting the elliptic distribution (3) onto the real (imaginary) axis before taking the limit $\tau \rightarrow \pm 1$.

It might be striking that the derivation of the (apparently simple) functional form of $\rho(z, \bar{z})$ for the Girko-Ginibre ensemble is less straightforward than the one for the (more complex) Wigner semicircle law. The reason is that there are many powerful methods invented for Hermitian random matrices: via orthogonal polynomials or Selberg's integral [5], supersymmetric method [6], diagrammatic expansion [7], Dyson gas [8] and free random variables [9].

In this paper we would like to present a result for non-Hermitian random matrices which is to a large extent universal, similarly to the two classical examples cited above. We shall show that the eigenvalue density $\rho_X(z, \bar{z})$ of a product

$$X = X_1 X_2 \cdots X_M, \quad (4)$$

of $M \geq 2$ independent $N \times N$ Gaussian matrices for which $\langle X_{1,ij} \rangle = \cdots = \langle X_{M,ij} \rangle = 0$ and $\langle |X_{1,ij}|^2 \rangle = \sigma_1^2/N, \dots, \langle |X_{M,ij}|^2 \rangle = \sigma_M^2/N$ for all i, j , assumes in the limit of $N \rightarrow \infty$ the following form:

$$\rho_X(z, \bar{z}) = \begin{cases} \frac{1}{M\pi} \sigma^{-\frac{2}{M}} |z|^{-2+\frac{2}{M}} & \text{for } |z| \leq \sigma, \\ 0 & \text{for } |z| > \sigma, \end{cases} \quad (5)$$

where the effective scale parameter $\sigma = \sigma_1 \sigma_2 \cdots \sigma_M$. This surprisingly simple formula is the main result of our paper. What is even more surprising is that this formula holds for a product of independent but not identically distributed Gaussian matrices. This means that the individual matrices X_i 's in the product may come from different Gaussian ensembles (GUE, GOE or various elliptic Gaussian non-Hermitian matrices) and the eigenvalue density will always be given by (5). In other words, even if X_1, \dots, X_M have oblate eigenvalue spectra, with $\tau_1 \neq 0, \dots, \tau_M \neq 0$, their product will have a rotationally-symmetric one. We shall derive this result with help of a diagrammatic technique appropriately tailored to non-Hermitian random matrices [10, 11] and to products of random matrices [12]. In order to make the paper self-contained we will also give an introduction to the diagrammatic methods (for a brief review see also [13]).

It is tempting to conjecture that the limit law for the product (5) holds also for a wider class of matrices, including Wigner matrices whose elements are independent, identically distributed random variables with a finite variance or more generally, for matrices which fulfill the Pastur-Lindeberg's condition [3]. We will present a numerical support for this conjecture.

The second objective of this paper is to use (5) in order to verify an interesting conjecture made in Ref. [14] saying that if the eigenvalue density $\rho(x, y)$ of a non-Hermitian matrix X is rotationally symmetric on the complex plane $z = x + iy$, then the marginal distribution $\rho_*(x) = \int dy \rho(x, y)$ obtained by its projection onto the real axis or a projection $\rho_*(y) = \int dx \rho(x, y)$ onto the imaginary axis must be equal to the eigenvalue density of the matrix $(X + X^\dagger)/\sqrt{8}$ or $i(X - X^\dagger)/\sqrt{8}$, respectively, both being Hermitian matrices. If true, this would allow one to calculate $\rho(x, y)$ from $\rho_*(x)$ via the inverse Abel transform. In particular, if one projects the Girko-Ginibre distribution (2) onto the real (or imaginary) axis, one indeed obtains the Wigner semicircle law: $\rho_*(x) = \frac{2}{\pi\sigma^2} \sqrt{\sigma^2 - x^2}$, which is the same as the eigenvalue density of the matrix $(X + X^\dagger)/\sqrt{8}$ (or $i(X - X^\dagger)/\sqrt{8}$). In [14] it was checked numerically that the relation seemed to apply also to more complicated ensembles. Here we shall present a counterexample by showing that the projection of the eigenvalue density of a product AB of two Hermitian matrices A and B which is rotationally symmetric (5) is different from the eigenvalue density of the rescaled anti-commutator $(AB + BA)/\sqrt{8}$ and the commutator $i(AB - BA)/\sqrt{8}$, so the conjecture is not true.

II. GENERALITIES

A. Eigenvalue density and the measure

We are interested in the eigenvalue distribution of a random matrix X (4) being a product of M independent $N \times N$ real or complex Gaussian matrices. The eigenvalues $\{\lambda_i\}$ of X are complex since X may be general be non-Hermitian.

The eigenvalue distribution is defined by

$$\rho_X(z, \bar{z}) = \left\langle \frac{1}{N} \sum_{i=1}^N \delta^{(2)}(z - \lambda_i) \right\rangle, \quad (6)$$

where \bar{z} denotes complex conjugate of z . The averaging $\langle \dots \rangle = \int \dots d\mu(X_1, \dots, X_M)$ is done with a factorized probability measure, which in the simplest case of identically distributed matrices takes the form

$$d\mu(X_1, \dots, X_M) \propto \prod_{\mu=1}^M e^{-\frac{N\alpha}{4} \text{Tr} X_\mu X_\mu^\dagger} DX_\mu, \quad (7)$$

where DX_μ denotes a flat measure. This formula applies to four generic cases of X_μ being (a) complex, (b) complex Hermitian, (c) real and (d) real symmetric matrices. The parameter α is defined as $\alpha = \lim_{N \rightarrow \infty} 2N_{\text{dof}}/N^2$ where N_{dof} is the number of real degrees of freedom of the matrix X . For (a) the flat measure is given by $DX_\mu = \prod_{ij} dX_{\mu,ij} d\bar{X}_{\mu,ij}$ or equivalently by $DX_\mu = \prod_{ij} d(\text{Re} X_{\mu,ij}) d(\text{Im} X_{\mu,ij})$ and $\alpha = 4$; for (b) $DX_\mu = \prod_i dX_{ii} \prod_{i>j} d(\text{Re} X_{\mu,ij}) d(\text{Im} X_{\mu,ij})$, $\alpha = 2$; for (c) $DX_\mu = \prod_{ij} dX_{\mu,ij}$, $\alpha = 2$ and finally for (d) $DX_\mu = \prod_{i \geq j} dX_{\mu,ij}$, $\alpha = 1$. For (c) and (d) the Hermitian conjugate X_μ^\dagger reduces to the transpose X_μ^T . The proportionality symbol in (7) means that the measure is displayed without a normalization constant which is fixed by the condition $\int d\mu(X_1, \dots, X_M) = 1$.

With this choice of α the variance of individual elements $\langle |X_{\mu,ij}|^2 \rangle = 1/N$ so that the scaling parameters $\sigma_1 = \dots = \sigma_M = 1$ and hence $\sigma = 1$ in Eq. (5). This means that the eigenvalue density of individual matrices X_μ is given by the Girko-Ginibre law (2) for (a) and (c) and the Wigner law (1) for (b) and (d), in both cases with $\sigma = 1$. For sake of simplicity we stick to this choice in the rest of the paper. The spectrum for arbitrary $\sigma_1, \dots, \sigma_M$ can be obtained by a trivial rescaling.

Later on we will also consider a general case of matrices from the elliptic ensemble with the eigenvalue distribution (3). We will also consider a product of non-identically distributed matrices, where X_1, \dots, X_M belong to different elliptic ensembles.

B. The Green's function

We shall follow here the standard strategy of calculating the eigenvalue density of a random matrix by first calculating the Green's function $g(z, \bar{z})$ and then using an exact relation between the eigenvalue density and the Green's function. Let us recall this relation. Using the following representation of the two-dimensional delta function

$$\delta^{(2)}(z - \lambda) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \frac{\epsilon^2}{(|z - \lambda|^2 + \epsilon^2)^2} = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \frac{\partial}{\partial \bar{z}} \left[\frac{\bar{z} - \bar{\lambda}}{|z - \lambda|^2 + \epsilon^2} \right], \quad (8)$$

one finds [4, 15–17] that

$$\rho_X(z, \bar{z}) = \frac{1}{\pi} \frac{\partial g(z, \bar{z})}{\partial \bar{z}}, \quad (9)$$

where

$$g(z, \bar{z}) = \lim_{\epsilon \rightarrow 0} \left\langle \frac{1}{N} \sum_i \frac{\bar{z} - \bar{\lambda}_i}{|z - \lambda_i|^2 + \epsilon^2} \right\rangle = \lim_{\epsilon \rightarrow 0} \left\langle \frac{1}{N} \text{Tr} \frac{\bar{z} \mathbb{1}_N - X^\dagger}{(\bar{z} \mathbb{1}_N - X^\dagger)(z \mathbb{1}_N - X) + \epsilon^2 \mathbb{1}_N} \right\rangle, \quad (10)$$

and $\mathbb{1}_N$ is an $N \times N$ identity matrix. As we shall see later, the Green's function can be calculated in the limit $N \rightarrow \infty$ using a summation method for planar Feynman diagrams. It is convenient to think of $g(z, \bar{z})$ as a part of a larger object [18], a $2N \times 2N$ matrix G with four $N \times N$ blocks [10, 11]:

$$G = \begin{pmatrix} G_{zz} & G_{z\bar{z}} \\ G_{\bar{z}z} & G_{\bar{z}\bar{z}} \end{pmatrix} = \lim_{\epsilon \rightarrow 0} \left\langle \begin{pmatrix} z \mathbb{1}_N - X & i\epsilon \mathbb{1}_N \\ i\epsilon \mathbb{1}_N & \bar{z} \mathbb{1}_N - X^\dagger \end{pmatrix}^{-1} \right\rangle. \quad (11)$$

Before we continue let us shortly comment on the notation used in the last formula, since we will also use it in the remaining part of the paper. The subscripts zz , $z\bar{z}$, $\bar{z}z$ and $\bar{z}\bar{z}$ refer to the position of the $N \times N$ blocks in the corresponding $2N \times 2N$ matrix. In the shorthand notation the arguments (z, \bar{z}) of a function defined on the complex

plane are skipped, so the correct reading of, for instance, G_{zz} is $G_{zz} = G_{zz}(z, \bar{z})$. We will also use a convention that the normalized trace of an $N \times N$ matrix denoted by a capital letter will be denoted by the corresponding small letter, for instance $g_{z\bar{z}} = \frac{1}{N} \text{Tr} G_{z\bar{z}}$.

Now coming back to the problem, by inverting the matrix in the brackets on the right-hand side in the last equation we can see that the Green's function $g(z, \bar{z})$ is equal to the normalized trace of the upper-left sub-matrix,

$$g(z, \bar{z}) \equiv g_{zz}(z, \bar{z}) = \frac{1}{N} \text{Tr} G_{zz}(z, \bar{z}). \quad (12)$$

When one calculates the Green's function (10) or the matrix G (11), one has to take the limit $N \rightarrow \infty$ first, and only then allow for $\epsilon \rightarrow 0$. This comes from the following reasoning. If $\epsilon = 0$, for finite N the function in the brackets $\langle \dots \rangle$ on the right hand side of (10) has isolated poles on the complex plane. However, in the limit $N \rightarrow \infty$ the poles coalesce and the function becomes non-holomorphic. One cannot then make an analytic continuation of the function from holomorphic to nonholomorphic region, as it is done when calculating G by diagrammatic method which utilizes $O(1/z)$ expansion. A small $\epsilon > 0$ is necessary to make G analytic everywhere. If one naively first took the limit $\epsilon \rightarrow 0$ and only then the limit $N \rightarrow \infty$, the matrix G would become block-diagonal: $G_{zz} = \langle (z - X)^{-1} \rangle$, $G_{z\bar{z}}^\dagger = \langle (\bar{z} - X^\dagger)^{-1} \rangle$ and $G_{z\bar{z}} = G_{\bar{z}z} = 0$. However, we shall see that

$$g_{z\bar{z}}(z, \bar{z}) = \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \left\langle \frac{1}{N} \text{Tr} \frac{-i\epsilon \mathbb{1}_N}{(\bar{z} \mathbb{1}_N - X^\dagger)(z \mathbb{1}_N - X) + \epsilon^2 \mathbb{1}_N} \right\rangle \quad (13)$$

and $g_{\bar{z}z}(z, \bar{z})$ differ from zero in the non-holomorphic region. In Ref. [10] it was shown that these quantities are related to the statistics of left and right eigenvectors of the non-Hermitian random matrix ensemble.

The quantities $g_{z\bar{z}} = g_{\bar{z}z}$ are purely imaginary, and $\gamma = -g_{z\bar{z}}g_{\bar{z}z}$ is a sort of order parameter for non-holomorphic behavior, which is positive in a region of the complex plane where the Green's function is non-holomorphic. The effect of pole coalescence and the emergence of a non-holomorphic behavior is very similar to the spontaneous breaking of a global symmetry in statistical models. In such systems the symmetry is preserved as long as the system size N is finite. It may, however, get spontaneously broken in the limit $N \rightarrow \infty$. Let us take the Ising model as an example. Its Hamiltonian is invariant under a global transformation flipping all spins and hence it has a Z_2 symmetry. As long as the number of spins is finite, the system is Z_2 -symmetric and the average magnetization, which is an order parameter, is equal zero. However in the thermodynamic limit, that is when the system size becomes infinite, the Z_2 symmetry gets spontaneously broken below a critical temperature and the average magnetization is non-zero. If one first calculated the average magnetization for a finite system and only then took the limit $N \rightarrow \infty$, the magnetization would be zero in this limit for all temperatures. To avoid the problem one can introduce a tiny external magnetic field h which weakly breaks the symmetry for finite-size systems. Now, if one first takes the limit $N \rightarrow \infty$ and only then $h \rightarrow 0$, one will obtain the correct result. In our case, the small parameter ϵ plays an analogous role to h and it guaranties that non-holomorphic contributions will be correctly picked up for $N \rightarrow \infty$.

C. Linearization

Let us have a closer look at the function in the brackets in the definition of the Green's function (10). In our original problem the matrix X is a product $X = X_1 \dots X_M$ of random matrices so it is a non-linear object from the point of view of the degrees of freedom that one has to average over. As a consequence the diagrammatic method would become very complicated. One can, however, linearize the problem by a trick used in [12] which relies on substituting X by a matrix Y of dimensions $MN \times MN$ which is linear in X_k 's and has eigenvalues closely related to those of X . The matrix Y is constructed from X_μ 's which are placed in a cyclic positions of a sparse $MN \times MN$ matrix,

$$Y = \begin{pmatrix} 0 & X_1 & & 0 \\ 0 & 0 & X_2 & 0 \\ & & \ddots & \ddots \\ 0 & & & 0 & X_{M-1} \\ X_M & & & & 0 \end{pmatrix}. \quad (14)$$

One can immediately discover a relation between eigenvalues of Y and those of $X = X_1 \dots X_M$ if one calculates the M -th power Y^M which gives a block-diagonal matrix

$$Y^M = \begin{pmatrix} Y_1 & & & 0 \\ & Y_2 & & \\ & & \ddots & \\ 0 & & & Y_M \end{pmatrix}, \quad (15)$$

with Y_μ being cyclic permutations of X_μ 's, $Y_\mu = X_\mu X_{\mu+1} \dots X_{\mu+M-1}$ (in the cyclic convention $X_{\mu+M} \equiv X_\mu$, and $X_0 \equiv X_M$). It is easy to see that all blocks Y_μ have the same eigenvalues. Indeed, if λ is an eigenvalue of Y_μ to an eigenvector \vec{v}_μ , $Y_\mu \vec{v}_\mu = \lambda \vec{v}_\mu$, it is also an eigenvalue of $Y_{\mu-1}$ to the eigenvector $\vec{v}_{\mu-1} = X_{\mu-1} \vec{v}_\mu$. One can see this by multiplying both sides $Y_\mu \vec{v}_\mu = \lambda \vec{v}_\mu$ by $X_{\mu-1}$, obtaining $X_{\mu-1} Y_\mu \vec{v}_\mu = \lambda X_{\mu-1} \vec{v}_\mu$ which is equivalent to $Y_{\mu-1} \vec{v}_{\mu-1} = \lambda \vec{v}_{\mu-1}$. In other words, the matrix Y^M has exactly the same eigenvalues as X and each eigenvalue is M -fold degenerated. Eigenvalues of X are thus related to those of Y as $\lambda_X = \lambda_Y^M$. The eigenvalue density $\rho_X(z, \bar{z})$ can be calculated from $\rho_Y(w, \bar{w})$ of Y by changing the variables $z = w^M$:

$$\rho_X(z, \bar{z}) = M \frac{\partial w}{\partial z} \frac{\partial \bar{w}}{\partial \bar{z}} \rho_Y(w, \bar{w}) = \frac{1}{M} |z|^{-2+\frac{2}{M}} \rho_Y(w(z), \bar{w}(\bar{z})). \quad (16)$$

The factor M in front of the Jacobian is related to the fact that the transformation $z = w^M$ maps the complex plane M times onto itself. The problem is thus reduced to finding the spectral density of Y , which is linear with respect to X_1, \dots, X_M . The density $\rho_Y(w, \bar{w})$ can be found from the appropriate Green's function. We will show below that $\rho_Y(w, \bar{w})$ is given by a Girko-Ginibre distribution (2), irrespectively of M and of $\tau_1, \tau_2 \dots$ and τ_M . This is a general result. In particular, for $M = 2$ the matrix Y (14) has an anti-diagonal block structure as chiral Gaussian matrices which have been intensively studied in the context of spectral properties of the Dirac operator in QCD [19]. In this case, the form of the eigenvalue density of Y for circular case ($\tau_1 = \tau_2 = 0$) can be inferred from results presented in [20–22] for complex, quaternion real, and real matrices, respectively.

III. GREEN'S FUNCTION AND PLANAR DIAGRAMS

In this section we recall the diagrammatic technique of calculating the Green's function. We begin with Hermitian matrices and later generalize the method to non-Hermitian ones and eventually to matrices which additionally have a block structure like the matrix Y from the previous section.

Let us make a general comment before we proceed. The diagrammatic method is based on the observation that the Green's function G can be interpreted as a generating function for connected two-point Feynman diagrams. In the limit $N \rightarrow \infty$ only planar diagrams contribute to G since non-planar ones are suppressed by at least a factor $O(1/N)$ [23, 24]. In this limit one can write a set of two self-consistent algebraic matrix equations which relate G to a generating function, Σ , for one-line irreducible diagrams. The equations are shown schematically in Fig. 1 and will be explained later. They can be solved for G . We want to stress that these equations have exactly the same form for Hermitian, complex matrices and for matrices with a block structure. They only differ by an algebraic structure reflecting indexing of the matrices G and Σ .

We finish with a remark that these equations hold for $N \rightarrow \infty$. In the context of the discussion about the order of taking the limits in (13) this means that one can safely set $\epsilon = 0$ since the limit $N \rightarrow \infty$ has already been taken.

A. Hermitian matrices

We will first demonstrate the diagrammatic technique on the example of Hermitian matrices and derive the Wigner semicircle law (1). Let us assume that $A = A^\dagger$, $A = \{A_{ab}\}$, $a = 1, \dots, N$, $b = 1, \dots, N$ is drawn from an ensemble with a probability measure

$$d\mu(A) \propto e^{-\frac{N}{2} \text{Tr} A^2} DA, \quad (17)$$

where $DA = \prod_a dA_{aa} \prod_{a>b} d(\text{Re} A_{ab}) d(\text{Im} A_{ab})$. The normalization constant, which is implicit in the above formula, is fixed by the condition $\int d\mu(A) = 1$. The eigenvalues λ_i of the matrix A are real. This makes the situation simpler than the one for general non-Hermitian matrices discussed in Sec. II. The eigenvalue density can be expressed as [1]

$$\rho(\lambda) = \left\langle \frac{1}{N} \sum_i^N \delta(\lambda - \lambda_i) \right\rangle, \quad (18)$$

where now the delta function is one-dimensional. Also the Green's function G matrix takes a simpler form,

$$G = \langle (Z - A)^{-1} \rangle \equiv \int (Z - A)^{-1} d\mu(A). \quad (19)$$

Here $Z = z\mathbb{1}_N$, where z is a complex number. The Green's function $g(z) \equiv \frac{1}{N}\text{Tr} G(z)$ is obtained by the Stieltjes transform of the eigenvalue density:

$$g(z) = \int d\lambda \frac{\rho(\lambda)}{z - \lambda}. \quad (20)$$

The last equation yields:

$$\rho(\lambda) = -\frac{1}{\pi} \text{Im} g(\lambda + i\epsilon), \quad (21)$$

for $\epsilon \rightarrow 0$, as follows from a standard representation of the one-dimensional delta function $\delta(x) = -\frac{1}{\pi} \text{Im}(x + i\epsilon)^{-1}$. The above Green's function can be calculated analytically in the large N limit, expanding (19) in terms of powers of Z^{-1} :

$$G(z) = Z^{-1} + \langle Z^{-1} A Z^{-1} A Z^{-1} \rangle + \langle Z^{-1} A Z^{-1} A Z^{-1} A Z^{-1} A Z^{-1} \rangle + \dots \quad (22)$$

Factors Z^{-1} are independent of A 's and thus can be pulled out of the average brackets. What remains are correlation functions of the type $\langle A_{i_1 i_2} \dots A_{i_{2n-1} i_{2n}} \rangle$ which by virtue of the Wick theorem can be expressed as products of two-point correlation functions (propagators)

$$\langle A_{ab} A_{cd} \rangle = \frac{1}{N} \delta_{ad} \delta_{bc}. \quad (23)$$

This observation allows one to graphically represent equation (22) as a sum over Feynman diagrams (see for instance [25]), as shown in Fig. 1B. Each propagator is represented as a double arc joining two pairs of matrix indices, while Z_{ab}^{-1} is drawn as a horizontal line joining indices a and b (Fig. 1A). In order to calculate G_{ab} one has to sum up contributions of all connected diagrams with two external points a, b . For finite N this is not an easy task because there are infinitely many diagrams. The problem enormously simplifies in the limit $N \rightarrow \infty$ since in this limit only planar diagrams contribute to the leading term of $1/N$ expansion and all non-planar diagrams can be neglected [23, 24]. It turns out that all planar diagrams can be summed up using an old trick known from field theory which reduces the problem to a closed set of equations for G . These equations are known as Dyson-Schwinger equations and we will discuss them now.

First, we introduce a generating function Σ for one-line irreducible diagrams, that is diagrams which cannot be split by cutting a single horizontal line (see Fig. 1C). Σ_{ab} generates all one-line irreducible diagrams with vertices a and b . The two generating functions are related to each other because any diagram from G can be constructed as a sandwich of horizontal lines and one-line irreducible diagrams (Fig. 1D):

$$G = Z^{-1} + Z^{-1} \Sigma Z^{-1} + Z^{-1} \Sigma Z^{-1} \Sigma Z^{-1} \Sigma Z^{-1} + \dots = (Z - \Sigma)^{-1}. \quad (24)$$

This matrix equation can be viewed as a definition of Σ . The introduction of Σ itself does not help to solve the problem. However, one can write down an independent equation for Σ and G . It follows from the observation that any one-line irreducible diagram can be obtained from a diagram from G by adding an arc (a propagator) to it (Fig. 1E). This gives

$$\Sigma_{ab} = \sum_{c,d} G_{cd} \frac{1}{N} \delta_{cd} \delta_{ab} = g \delta_{ab}, \quad (25)$$

or, in matrix notation, $\Sigma = g\mathbb{1}_N$. Taking trace of both sides we obtain $\sigma = g$ where $\sigma \equiv \frac{1}{N} \text{Tr} \Sigma$ is the normalized trace of Σ . The two equations (24) and (25) form a closed set of equations which can be solved for the Green's function $g(z)$. Inserting the last equation to (24) with $Z = z\mathbb{1}_N$ we have $g(z - g) = 1$ and hence $g(z) = \frac{1}{2}(z - \sqrt{z^2 - 4})$ and $\rho(\lambda) = \frac{1}{2\pi} \sqrt{4 - \lambda^2}$, as follows from (21).

B. Complex matrices

Let us now discuss how to calculate the Green's function in case of non-Hermitian Gaussian random matrices with complex entries (see for instance [13]). The probability measure is now

$$d\mu(A) \propto e^{-N \text{Tr} A A^\dagger} \prod_{i,j} d(\text{Re} A_{ij}) d(\text{Im} A_{ij}), \quad (26)$$

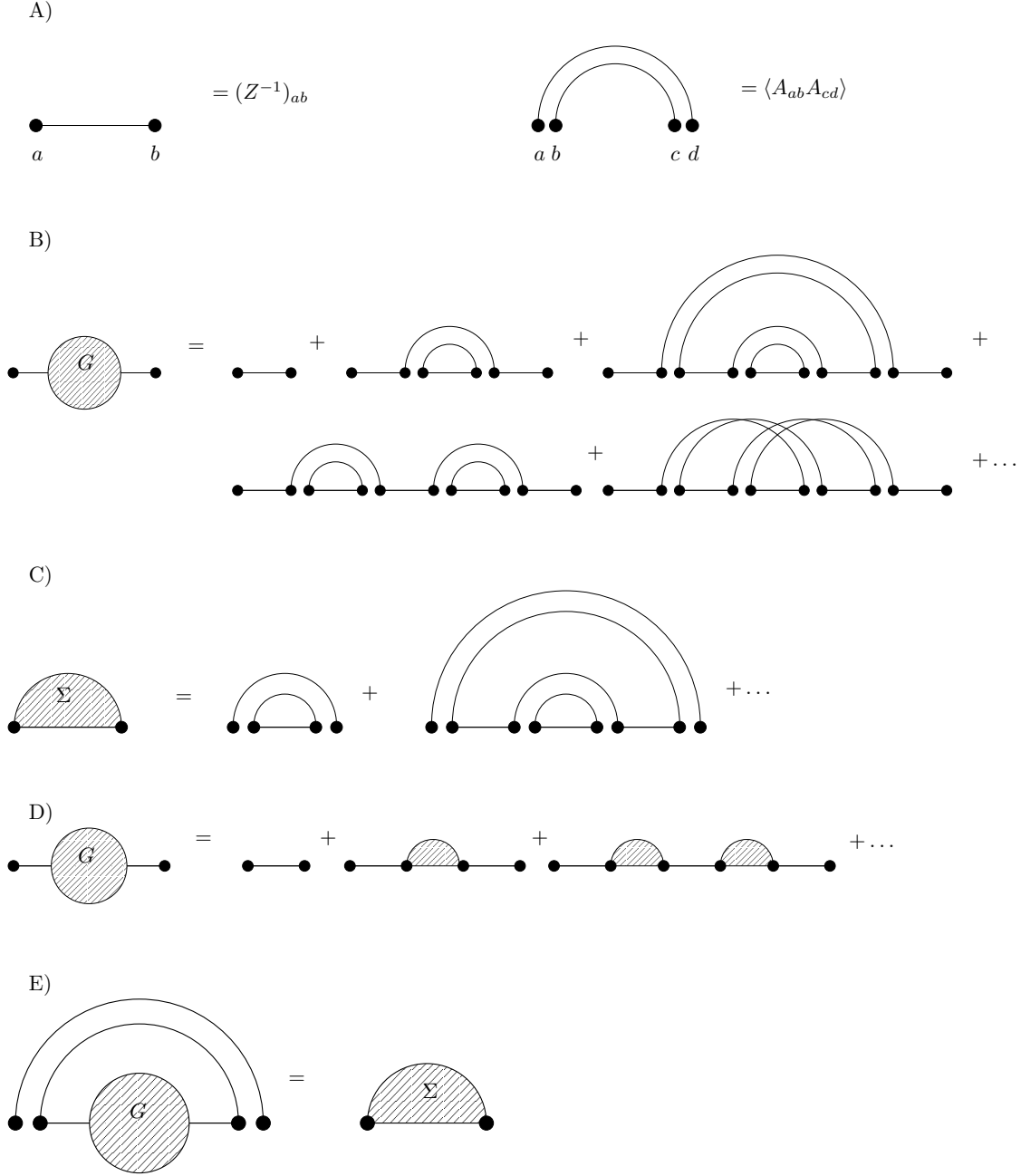


FIG. 1: (A) Feynman rules. $(Z^{-1})_{ab}$ is drawn as a line between a and b and the propagator $\langle A_{ab} A_{cd} \rangle$ as a double arc joining a with d and b with c , respectively. (B) Graphical representation of Eq. (22). The last three displayed graphs correspond to the third term in (22). The contribution of the last diagram can be neglected in the large N limit since it is non-planar and has a suppressing factor $1/N^2$. (C) Definition of self-energy Σ . (D) The first Dyson-Schwinger equation which relates G to Σ . (E) The second Dyson-Schwinger equation.

which corresponds to $\alpha = 4$ in Eq. (7). The propagators are

$$\begin{aligned}
 \langle A_{ab} A_{cd} \rangle &= 0 & , & \quad \langle A_{ab} A_{cd}^\dagger \rangle = \frac{1}{N} \delta_{ad} \delta_{bc}, \\
 \langle A_{ab}^\dagger A_{cd} \rangle &= \frac{1}{N} \delta_{ad} \delta_{bc} & , & \quad \langle A_{ab}^\dagger A_{cd}^\dagger \rangle = 0.
 \end{aligned}
 \tag{27}$$

It is convenient to think of A and A^\dagger as $N \times N$ sub-matrices of a $2N \times 2N$ matrix

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{zz} & \mathcal{A}_{z\bar{z}} \\ \mathcal{A}_{\bar{z}z} & \mathcal{A}_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^\dagger \end{pmatrix}. \quad (28)$$

The off-diagonal blocks are equal zero for this particular matrix. We use a convention discussed in Section II: the position of an $N \times N$ sub-matrix is denoted by subscripts z, \bar{z} . We apply the same notation to other $2N \times 2N$ matrices: the Green's function, the self-energy Σ and the matrix Z ,

$$G = \begin{pmatrix} G_{zz} & G_{z\bar{z}} \\ G_{\bar{z}z} & G_{\bar{z}\bar{z}} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{zz} & \Sigma_{z\bar{z}} \\ \Sigma_{\bar{z}z} & \Sigma_{\bar{z}\bar{z}} \end{pmatrix}, \quad Z = \begin{pmatrix} Z_{zz} & Z_{z\bar{z}} \\ Z_{\bar{z}z} & Z_{\bar{z}\bar{z}} \end{pmatrix}. \quad (29)$$

Matrix elements of the block G_{zz} of G will be denoted by G_{ab} , elements of $G_{z\bar{z}}$ by $G_{a\bar{b}}$, etc. In other words, the subscripts z and \bar{z} serve also as templates for the corresponding barred or unbarred indices. For completeness let us rewrite the propagators (27) using this notation:

$$\begin{aligned} \langle \mathcal{A}_{ab} \mathcal{A}_{cd} \rangle &= 0, & \langle \mathcal{A}_{ab} \mathcal{A}_{\bar{c}\bar{d}} \rangle &= \frac{1}{N} \delta_{a\bar{d}} \delta_{b\bar{c}}, \\ \langle \mathcal{A}_{\bar{a}\bar{b}} \mathcal{A}_{cd} \rangle &= \frac{1}{N} \delta_{\bar{a}d} \delta_{\bar{b}c}, & \langle \mathcal{A}_{\bar{a}\bar{b}} \mathcal{A}_{\bar{c}\bar{d}} \rangle &= 0. \end{aligned} \quad (30)$$

Now we are ready to write down Dyson-Schwinger equations for complex matrices. The first equation is identical to Eq. (24), except that now G , Σ and Z have dimensions $2N \times 2N$:

$$\begin{pmatrix} G_{zz} & G_{z\bar{z}} \\ G_{\bar{z}z} & G_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} Z_{zz} - \Sigma_{zz} & Z_{z\bar{z}} - \Sigma_{z\bar{z}} \\ Z_{\bar{z}z} - \Sigma_{\bar{z}z} & Z_{\bar{z}\bar{z}} - \Sigma_{\bar{z}\bar{z}} \end{pmatrix}^{-1}. \quad (31)$$

This equation is general, but later we will write it for a specific form of Z relevant for the calculation of the eigenvalue density. The second equation, which corresponds to (25), can be derived using the propagators defined in Eq. (30). It can be done separately in each sector zz , $z\bar{z}$, $\bar{z}z$ and $\bar{z}\bar{z}$:

$$\begin{aligned} \Sigma_{ad} &= 0, & \Sigma_{a\bar{d}} &= \frac{1}{N} \delta_{a\bar{d}} \delta_{b\bar{c}} G_{b\bar{c}} = \delta_{a\bar{d}} g_{z\bar{z}}, \\ \Sigma_{\bar{a}d} &= \frac{1}{N} \delta_{\bar{a}d} \delta_{\bar{b}c} G_{bc} = \delta_{\bar{a}d} g_{\bar{z}z}, & \Sigma_{\bar{a}\bar{d}} &= 0, \end{aligned} \quad (32)$$

where $g_{z\bar{z}} = \frac{1}{N} \text{Tr} G_{z\bar{z}}$ and $g_{\bar{z}z} = \frac{1}{N} \text{Tr} G_{\bar{z}z}$. In matrix notation the last equation can be written as

$$\begin{pmatrix} \Sigma_{zz} & \Sigma_{z\bar{z}} \\ \Sigma_{\bar{z}z} & \Sigma_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} 0 & g_{z\bar{z}} \mathbb{1}_N \\ g_{\bar{z}z} \mathbb{1}_N & 0 \end{pmatrix}. \quad (33)$$

One should note that the form of this equation is independent of Z while the form of the first Dyson-Schwinger equation (31) is independent of the propagator structure. If we insert now

$$Z = \lim_{\epsilon \rightarrow 0} \begin{pmatrix} z \mathbb{1}_N & i\epsilon \mathbb{1}_N \\ i\epsilon \mathbb{1}_N & \bar{z} \mathbb{1}_N \end{pmatrix} = \begin{pmatrix} z \mathbb{1}_N & 0 \\ 0 & \bar{z} \mathbb{1}_N \end{pmatrix} \quad (34)$$

to Eq. (31), remembering that we are allowed to take $\epsilon \rightarrow 0$ since all above equations are derived for large N and hence the limit $N \rightarrow \infty$ has been taken, we eventually obtain a matrix equation

$$\begin{pmatrix} G_{zz} & G_{z\bar{z}} \\ G_{\bar{z}z} & G_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} z \mathbb{1}_N - \Sigma_{zz} & -\Sigma_{z\bar{z}} \\ -\Sigma_{\bar{z}z} & \bar{z} \mathbb{1}_N - \Sigma_{\bar{z}\bar{z}} \end{pmatrix}^{-1}, \quad (35)$$

which together with (33) forms a closed set of algebraic equations for $G(z, \bar{z})$.

We will now solve this set of equations and then determine $\rho(z, \bar{z})$ using Eq. (9). We first notice that Eq. (33) reduces to a 2×2 matrix equation:

$$\begin{pmatrix} \sigma_{zz} & \sigma_{z\bar{z}} \\ \sigma_{\bar{z}z} & \sigma_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} 0 & g_{z\bar{z}} \\ g_{\bar{z}z} & 0 \end{pmatrix}, \quad (36)$$

where, as before, small letters denote the normalized traces of the corresponding blocks, for instance $\sigma_{zz} = \frac{1}{N} \text{Tr} \Sigma_{zz}$. Similarly, equation (35) reduces to

$$\begin{pmatrix} g_{zz} & g_{z\bar{z}} \\ g_{\bar{z}z} & g_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} z - \sigma_{zz} & -\sigma_{z\bar{z}} \\ -\sigma_{\bar{z}z} & \bar{z} - \sigma_{\bar{z}\bar{z}} \end{pmatrix}^{-1}, \quad (37)$$

which, after eliminating σ 's with help of Eq. (36), leads to

$$\begin{pmatrix} g_{zz} & g_{z\bar{z}} \\ g_{\bar{z}z} & g_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} z & -g_{z\bar{z}} \\ -g_{\bar{z}z} & \bar{z} \end{pmatrix}^{-1} = \frac{1}{|z|^2 - g_{z\bar{z}}g_{\bar{z}z}} \begin{pmatrix} \bar{z} & g_{z\bar{z}} \\ g_{\bar{z}z} & z \end{pmatrix}. \quad (38)$$

This equation has two solutions. The first one corresponds to $g_{\bar{z}z} = g_{z\bar{z}} = 0$ which gives $g_{zz} = z^{-1}$ and is equivalent to the trivial holomorphic solution and hence must be true for large $|z|$. The second solution corresponds to $|z|^2 - g_{z\bar{z}}g_{\bar{z}z} = 1$. In this case the off-diagonal blocks are different from zero and $g_{zz} = \bar{z}$. The two solutions match for $|z|^2 = 1$. Therefore, the first solution holds outside the unit circle and the second one inside the circle. Using the Gauss law (9) one finds

$$\rho(z, \bar{z}) = \begin{cases} \frac{1}{\pi} & \text{for } |z| \leq 1, \\ 0 & \text{for } |z| > 1, \end{cases} \quad (39)$$

which is the celebrated Girko-Ginibre distribution [15, 16].

To summarize this part, one can write the closed set of algebraic equations for G and Σ in the large- N limit using diagrammatic relations between the generating function for connected two-point planar diagrams (given by G) and the generating function for one-line irreducible two-point planar diagrams (given by the free energy Σ). One can set $\epsilon = 0$ in these equations since they are derived already in the limit $N \rightarrow \infty$.

C. Complex matrices with a block structure

We are now ready to calculate the Green's function $g_Y(w, \bar{w})$ for the matrix Y (14) which has blocks X_μ being independent complex non-Hermitian Gaussian matrices [12]. The matrix G will be now a $2NM \times 2NM$ matrix having four $NM \times NM$ blocks G_{ww} , $G_{w\bar{w}}$, $G_{\bar{w}w}$ and $G_{\bar{w}\bar{w}}$ which themselves consists of M^2 blocks of size $N \times N$ which we shall denote by $G_{\mu\nu}$, $G_{\mu\bar{\nu}}$, $G_{\bar{\mu}\nu}$ and $G_{\bar{\mu}\bar{\nu}}$ respectively, for instance

$$G_{w\bar{w}} = \begin{pmatrix} G_{1\bar{1}} & \dots & G_{1\bar{M}} \\ & \ddots & \\ G_{M\bar{1}} & \dots & G_{M\bar{M}} \end{pmatrix}. \quad (40)$$

There is an analogous block structure for the matrix Σ . One should distinguish Greek subscripts from Latin subscripts giving the position of the matrix elements within the block. For instance, $\Sigma_{\mu\bar{\nu}}$ is an $N \times N$ sub-matrix of the block $\Sigma_{w\bar{w}}$ and $(\Sigma_{\mu\bar{\nu}})_{a\bar{b}}$ is an element of this sub-matrix. In this convention the normalized trace of a block is $\sigma_{\mu\bar{\nu}} = \frac{1}{N} \text{Tr} \Sigma_{\mu\bar{\nu}} = \frac{1}{N} \sum_{a=1}^N (\Sigma_{\mu\bar{\nu}})_{a\bar{a}}$. One can now repeat the same procedure which we applied to the matrix having a single block and derive exact relations between the generating function G and Σ in the planar limit. The first Dyson-Schwinger equation,

$$\begin{pmatrix} G_{ww} & G_{w\bar{w}} \\ G_{\bar{w}w} & G_{\bar{w}\bar{w}} \end{pmatrix} = \begin{pmatrix} w\mathbb{1}_{NM} - \Sigma_{ww} & -\Sigma_{w\bar{w}} \\ -\Sigma_{\bar{w}w} & \bar{w}\mathbb{1}_{NM} - \Sigma_{\bar{w}\bar{w}} \end{pmatrix}^{-1}, \quad (41)$$

is almost identical as (35), except that the blocks and the identity matrices are now of dimensions $NM \times NM$. To write the second equation, we need to know the propagators. Let us first define a $2NM \times 2NM$ matrix \mathcal{Y} , a counterpart of \mathcal{A} from Eq. (28):

$$\mathcal{Y} = \begin{pmatrix} \mathcal{Y}_{ww} & \mathcal{Y}_{w\bar{w}} \\ \mathcal{Y}_{\bar{w}w} & \mathcal{Y}_{\bar{w}\bar{w}} \end{pmatrix} = \begin{pmatrix} Y & 0 \\ 0 & Y^\dagger \end{pmatrix}, \quad (42)$$

where Y is cyclic as defined in Eq. (14) and Y^\dagger is anti-cyclic,

$$Y^\dagger = \begin{pmatrix} 0 & & & & X_M^\dagger \\ X_1^\dagger & 0 & & & 0 \\ & \ddots & & & \\ & & X_{M-2}^\dagger & 0 & \\ 0 & & & X_{M-1}^\dagger & 0 \end{pmatrix}. \quad (43)$$

Since the block matrices $\mathcal{Y}_{\mu\mu+1} = X_\mu$ are assumed to be independent of each other, the only non-zero propagators are

$$\langle \mathcal{Y}_{12,ab} \mathcal{Y}_{2\bar{1},\bar{c}\bar{d}} \rangle = \langle \mathcal{Y}_{23,ab} \mathcal{Y}_{3\bar{2},\bar{c}\bar{d}} \rangle = \dots = \langle \mathcal{Y}_{M1,ab} \mathcal{Y}_{1\bar{M},\bar{c}\bar{d}} \rangle = \frac{1}{N} \delta_{a\bar{d}} \delta_{b\bar{c}}, \quad (44)$$

or in short

$$\langle \mathcal{Y}_{12} \mathcal{Y}_{2\bar{1}} \rangle = \langle \mathcal{Y}_{23} \mathcal{Y}_{3\bar{2}} \rangle = \dots = \langle \mathcal{Y}_{M1} \mathcal{Y}_{1\bar{M}} \rangle = \mathbb{T}, \quad (45)$$

where the tensor \mathbb{T} has elements $T_{abcd} = \frac{1}{N} \delta_{ab} \delta_{cd}$, with indices corresponding to the those of the matrices on the left-hand side. If we now insert these propagators to the second Dyson-Schwinger equation, we obtain

$$\Sigma_{\mu\bar{\mu}} = g_{\mu+1\bar{\mu}+1} \mathbb{1}_N, \quad (46)$$

and $\Sigma_{\mu\bar{\nu}} = \Sigma_{\bar{\mu}\nu} = 0$ for $\mu \neq \nu$. The problem is symmetric with respect to permutation of the matrices X_μ , so $g_{1\bar{1}} = \dots = g_{M\bar{M}} \equiv g_{w\bar{w}}$ in the whole $w\bar{w}$ -block and similarly in the $\bar{w}w$ -block. Thus the last equation can be compactly written as

$$\Sigma_{w\bar{w}} = g_{w\bar{w}} \mathbb{1}_{NM}, \quad \Sigma_{\bar{w}w} = g_{\bar{w}w} \mathbb{1}_{NM}, \quad (47)$$

where $\mathbb{1}_{NM}$ is now the identity $NM \times NM$ matrix for the whole block, $g_{w\bar{w}} = \frac{1}{NM} \text{Tr} G_{w\bar{w}}$ and $g_{\bar{w}w} = \frac{1}{NM} \text{Tr} G_{\bar{w}w}$. Inserting $\Sigma_{ww} = \Sigma_{\bar{w}\bar{w}} = 0$ and (47) to (41) we see that each block on the right-hand side of (41) is proportional to the identity matrix. Thus equation (41) reduces to a 2×2 matrix equation for the normalized traces which play the role of proportionality coefficients at the identity matrices,

$$\begin{pmatrix} g_{ww} & g_{w\bar{w}} \\ g_{\bar{w}w} & g_{\bar{w}\bar{w}} \end{pmatrix} = \begin{pmatrix} w & -g_{w\bar{w}} \\ -g_{\bar{w}w} & \bar{w} \end{pmatrix}^{-1}. \quad (48)$$

This is identical to (38) for a complex matrix with a single block discussed in the previous section. In other words, the Green's function and hence also the eigenvalue density of the matrix Y does not depend on the number of blocks in Y and is given by the Girko-Ginibre law [15, 16]

$$\rho_Y(w, \bar{w}) = \begin{cases} \frac{1}{\pi} & \text{for } |w| \leq 1, \\ 0 & \text{for } |w| > 1. \end{cases} \quad (49)$$

This result is valid also for other matrices considered in Eq. (7), that is for real non-symmetric and Hermitian complex matrices, as long as $M > 1$. It is so because what matters is the structure of propagators only, which is the same for all mentioned ensembles. In particular, for $M = 2$ one can deduce this formula from considerations of chiral ensembles [20–22]. In the next section we shall show how to derive the above result for the product of M elliptic complex and/or real matrices with different oblateness parameters $\tau_1 \neq \dots \neq \tau_M$. Now we will only observe that by inserting the Girko-Ginibre spectrum into Eq. (16) we finally obtain

$$\rho_X(z, \bar{z}) = \rho_X(|z|) = \begin{cases} \frac{1}{M\pi} |z|^{-2+\frac{2}{M}} & \text{for } |z| \leq 1, \\ 0 & \text{for } |z| > 1, \end{cases} \quad (50)$$

which completes the derivation of our main result. In Figs. 2 and 3 we show a comparison between the above formula and the spectrum of X obtained numerically by diagonalization of finite matrices. The agreement is very good. For the spectrum of the product of two Hermitian matrices (GUE) shown in the left panel of Fig. 2 we observe a small deviation from rotational symmetry manifesting as an accumulation of eigenvalues along the real axis and a depletion of eigenvalues in a narrow strip close to this axis. The number of eigenvalues on the axis grows as \sqrt{N} and the width of the strip decreases as $1/\sqrt{N}$ when $N \rightarrow \infty$. This effect is almost identical as the one known for real Girko-Ginibre matrices [26, 27]. If one multiplies three or more GUE matrices the effect disappears. A difference between the product of two and the product of more than two GUE matrices is that for two the trace $\text{Tr} X_1 X_2$ is real whereas for three (or more) it is not. In other words, the constraint of the trace to be real introduces a weak spherical symmetry breaking of the eigenvalue spectrum.

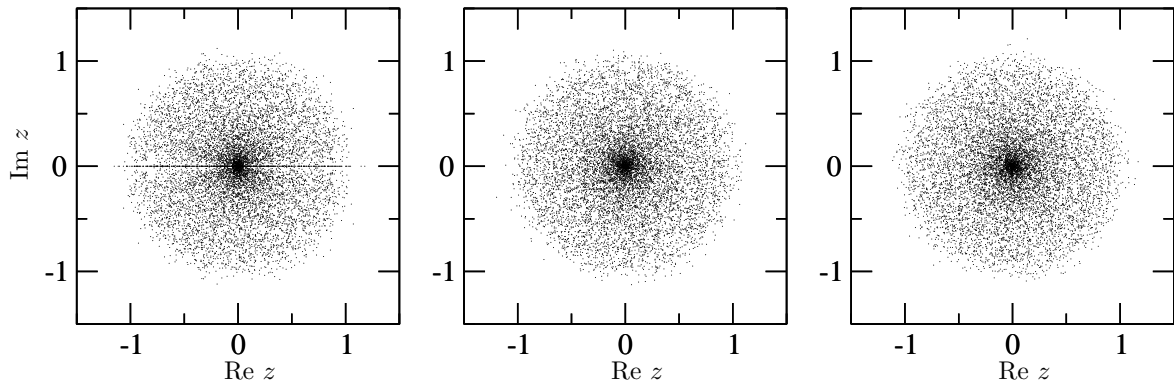


FIG. 2: Plots of $\rho_X(z, \bar{z})$ for X_1, X_2 being two Hermitian matrices (left), two complex matrices (middle), and for X_1 being a Hermitian and X_2 an elliptic random matrix with $\phi = \pi/3$ (right). For each case 100 matrices of size $N = 100$ were generated.

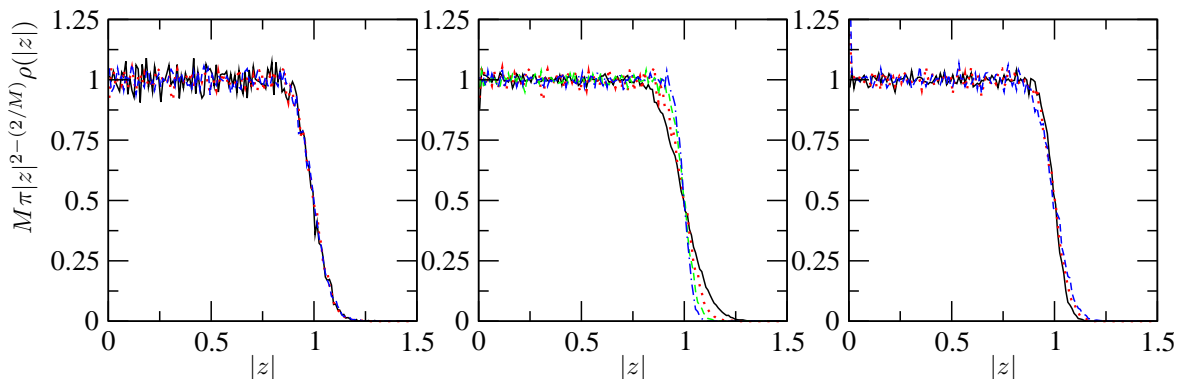


FIG. 3: Plots of $M\pi|z|^{2-\frac{2}{M}}\rho_X(|z|)$ obtained from simulations for various M and matrix sizes N . The theoretical distribution (not shown in the figure) which corresponds to (5) is a step function $f(|z|) = 1$ for $0 < |z| < 1$ and zero otherwise. Left: $X = X_1 X_2$ ($M = 2$) for $N = 100$ and X_1, X_2 taken from the same ensembles as in Fig. 2: black solid line for Hermitian, red dotted line for complex, and blue dashed line for Hermitian elliptic matrices. Middle: $M = 2$, complex matrices of size $N = 50, 100, 200, 400$ (black solid, red dotted, green dashed and blue dotted-dashed lines, respectively). To obtain these plots, we averaged spectra of 10000, 1000, 1000 and 500 matrices and constructed histograms of absolute values of their eigenvalues. Right: $N = 200$ and $M = 2, 3, 4$ (black solid, red dotted and blue dashed lines). For each M , 1000 matrices were generated.

IV. PRODUCT OF ARBITRARY GAUSSIAN MATRICES (ELLIPTIC ENSEMBLES)

Let us now consider a general class of non-Hermitian random matrices which include as special cases the well known examples of Hermitian (GUE), Girko-Ginibre, and anti-Hermitian ensembles. These “elliptic” ensembles were first introduced in [4] and can be defined as follows. A complex, elliptic matrix X is obtained as a linear combination of two identical, independent Hermitian Gaussian matrices A, B : $X = \cos(\phi)A + i \sin(\phi)B$, mixed with an arbitrary real mixing parameter ϕ . Since A and B are independent, the corresponding propagators are $\langle A_{ab}A_{cd} \rangle = \frac{1}{N}\delta_{ad}\delta_{bc}$, $\langle B_{ab}B_{cd} \rangle = \frac{1}{N}\delta_{ad}\delta_{bc}$, and $\langle A_{ab}B_{cd} \rangle = 0$. When one changes variables from A and B to X and X^\dagger one finds

$$\langle X_{ab}X_{cd} \rangle = \langle X_{ab}^\dagger X_{cd}^\dagger \rangle = \tau \cdot \frac{1}{N}\delta_{ad}\delta_{bc}, \quad \langle X_{ab}X_{cd}^\dagger \rangle = \langle X_{ab}^\dagger X_{cd} \rangle = \frac{1}{N}\delta_{ad}\delta_{bc}, \quad (51)$$

where $\tau = \cos(2\phi)$. The corresponding integration measure for X reads:

$$d\mu(X) \propto \exp \left\{ -N \frac{1}{1-\tau^2} \left(\text{Tr} X X^\dagger - \tau \frac{1}{2} \text{Tr} (X X + X^\dagger X^\dagger) \right) \right\} \prod_{ij} d(\text{Re} X_{ij}) d(\text{Im} X_{ij}). \quad (52)$$

For $\phi = 0$ ($\tau = 1$) the matrix X is Hermitian, for $\phi = \pi/2$ ($\tau = -1$) it is anti-Hermitian while for $\phi = \pi/4$ ($\tau = 0$) it is isotropic complex.

A. Eigenvalue distribution of a single elliptic random matrix

One can determine the eigenvalue distribution of X using the same methods as in Sec. III B. The only difference is that the propagators $\langle X_{ab} X_{cd} \rangle = \langle X_{ab}^\dagger X_{cd}^\dagger \rangle$ (51) do not vanish but are proportional to τ . This leads to the following modification of the first Dyson-Schwinger equation (36):

$$\begin{pmatrix} \sigma_{zz} & \sigma_{z\bar{z}} \\ \sigma_{\bar{z}z} & \sigma_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} \tau g_{zz} & g_{z\bar{z}} \\ g_{\bar{z}z} & \tau g_{\bar{z}\bar{z}} \end{pmatrix}, \quad (53)$$

while the second one (37) stays intact:

$$\begin{pmatrix} g_{zz} & g_{z\bar{z}} \\ g_{\bar{z}z} & g_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} z - \sigma_{zz} & -\sigma_{z\bar{z}} \\ -\sigma_{\bar{z}z} & \bar{z} - \sigma_{\bar{z}\bar{z}} \end{pmatrix}^{-1}. \quad (54)$$

These equations can be solved for g_{zz} . The solution reads

$$g_{zz} = \begin{cases} \frac{\bar{z} - \tau z}{1 - \tau^2} & \text{for } \frac{x^2}{(1+\tau)^2} + \frac{y^2}{(1-\tau)^2} \leq 1, \\ \frac{z - \sqrt{z^2 - 4\tau}}{2\tau} & \text{otherwise,} \end{cases} \quad (55)$$

where $z = x + iy$. The non-holomorphic solution matches the holomorphic one on the ellipse. The eigenvalue density is [4]

$$\rho(z, \bar{z}) = \frac{1}{\pi} \frac{\partial g_{zz}}{\partial \bar{z}} = \begin{cases} \frac{1}{\pi(1-\tau^2)} & \text{for } \frac{x^2}{(1+\tau)^2} + \frac{y^2}{(1-\tau)^2} \leq 1, \\ 0 & \text{otherwise.} \end{cases} \quad (56)$$

The parameter τ is a measure of flattening of the ellipse on which $\rho(z, \bar{z}) > 0$. For $\tau = 0$ the last equation reproduces the result for non-Hermitian complex matrices. For $\tau \rightarrow 1$, the ellipse reduces to a cut on the real axis: $\rho_*(x) = \int dy \rho(x, y)$, and then take the limit $\tau \rightarrow 1$. One recovers the Wigner semicircle law $\rho_*(x) = \frac{1}{2\pi} \sqrt{4 - x^2}$, as expected.

B. Eigenvalue distribution of a product of two or more elliptic random matrices

We are now interested in the eigenvalue density of the product (4) where X_μ 's are drawn from a Gaussian ensemble with the measure (52). We shall show that the result is again given by Eq. (5) and hence exhibits a large degree of universality: it does not depend on τ and is exactly the same even if each of the matrices X_μ is drawn from a Gaussian ensemble with a different flattening parameter τ_μ . We will derive (5) for $X = X_1 X_2$ and then make a comment on the generalization to $M > 2$.

We will use the linearization and calculate first the eigenvalue density of the matrix Y (14) constructed from X_1 and X_2 , having the only non-vanishing propagators given by (51) with two parameters τ_1 and τ_2 . As before, first we have to determine the propagator structure for the block matrix \mathcal{Y} (42) and then apply it to derive the Dyson-Schwinger equation. The matrix \mathcal{Y} reads

$$\mathcal{Y} = \begin{pmatrix} Y & 0 \\ 0 & Y^\dagger \end{pmatrix} = \begin{pmatrix} 0 & X_1 & 0 & 0 \\ X_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_2^\dagger \\ 0 & 0 & X_1^\dagger & 0 \end{pmatrix}. \quad (57)$$

The first non-vanishing propagator comes from the correlations between X_μ 's and X_μ^\dagger 's, exactly as in Eq. (45):

$$\langle \mathcal{Y}_{12} \mathcal{Y}_{2\bar{1}} \rangle = \langle \mathcal{Y}_{21} \mathcal{Y}_{1\bar{2}} \rangle = \mathbb{T}. \quad (58)$$

The next one comes from autocorrelations of X_μ 's (51) which are proportional to τ ,

$$\langle \mathcal{Y}_{12} \mathcal{Y}_{12} \rangle = \tau_1 \mathbb{T}, \quad \langle \mathcal{Y}_{21} \mathcal{Y}_{21} \rangle = \tau_2 \mathbb{T}, \quad (59)$$

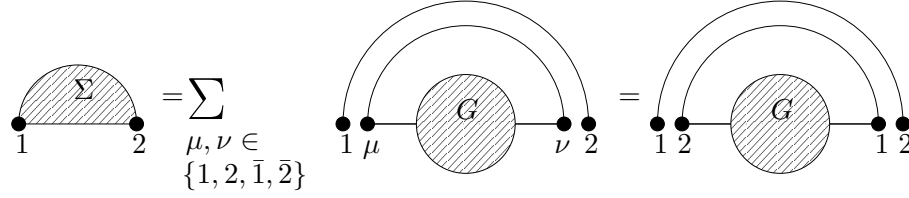


FIG. 4: Example of calculation of σ_{12} in Eq. (61). We write the second Dyson-Schwinger equation for Σ_{12} . The only non-vanishing propagator is the one between indices 1, 2 and $1, \bar{2}$. Taking the trace of both sides of the equation we arrive at $\sigma_{12} = \tau_1 g_{21}$.

and the last one from autocorrelations of X_μ^\dagger 's

$$\langle \mathcal{Y}_{1\bar{2}} \mathcal{Y}_{1\bar{2}} \rangle = \tau_1 \mathbb{T}, \quad \langle \mathcal{Y}_{2\bar{1}} \mathcal{Y}_{2\bar{1}} \rangle = \tau_2 \mathbb{T}. \quad (60)$$

Here \mathbb{T} denotes again a tensor with elements $T_{abcd} = \frac{1}{N} \delta_{ad} \delta_{bc}$, where a, b are indices of the first matrix and c, d of the second one on the right-hand sides of the above equations. All other correlations between the blocks of \mathcal{Y} vanish. We can now write two Dyson-Schwinger equations:

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{1\bar{1}} & \sigma_{1\bar{2}} \\ \sigma_{21} & \sigma_{22} & \sigma_{2\bar{1}} & \sigma_{2\bar{2}} \\ \sigma_{\bar{1}1} & \sigma_{\bar{1}2} & \sigma_{\bar{1}\bar{1}} & \sigma_{\bar{1}\bar{2}} \\ \sigma_{\bar{2}1} & \sigma_{\bar{2}2} & \sigma_{\bar{2}\bar{1}} & \sigma_{\bar{2}\bar{2}} \end{pmatrix} = \begin{pmatrix} 0 & \tau_1 g_{21} & g_{2\bar{2}} & 0 \\ \tau_2 g_{12} & 0 & 0 & g_{1\bar{1}} \\ g_{2\bar{2}} & 0 & 0 & \tau_1 g_{2\bar{1}} \\ 0 & g_{1\bar{1}} & \tau_2 g_{1\bar{2}} & 0 \end{pmatrix}, \quad (61)$$

and

$$\begin{pmatrix} g_{11} & g_{12} & g_{1\bar{1}} & g_{1\bar{2}} \\ g_{21} & g_{22} & g_{2\bar{1}} & g_{2\bar{2}} \\ g_{\bar{1}1} & g_{\bar{1}2} & g_{\bar{1}\bar{1}} & g_{\bar{1}\bar{2}} \\ g_{\bar{2}1} & g_{\bar{2}2} & g_{\bar{2}\bar{1}} & g_{\bar{2}\bar{2}} \end{pmatrix} = \begin{pmatrix} w - \sigma_{11} & -\sigma_{12} & -\sigma_{1\bar{1}} & -\sigma_{1\bar{2}} \\ -\sigma_{21} & w - \sigma_{22} & -\sigma_{2\bar{1}} & -\sigma_{2\bar{2}} \\ -\sigma_{\bar{1}1} & -\sigma_{\bar{1}2} & \bar{w} - \sigma_{\bar{1}\bar{1}} & -\sigma_{\bar{1}\bar{2}} \\ -\sigma_{\bar{2}1} & -\sigma_{\bar{2}2} & -\sigma_{\bar{2}\bar{1}} & \bar{w} - \sigma_{\bar{2}\bar{2}} \end{pmatrix}^{-1}. \quad (62)$$

In the first equation the off-diagonal blocks are the same as in the previous section (46). The diagonal blocks $\sigma_{ww}, \sigma_{\bar{w}\bar{w}}$ now depend on τ_μ 's. As an illustration we show in Fig. 4 a graphical representation of the equation for $\sigma_{12} = \tau_1 g_{21}$ which explains the flip of indices. Let us first look for a holomorphic solution, so assume that off-diagonal blocks of g vanish: $g_{w\bar{w}} = g_{\bar{w}w} = 0$. In this case the above equations reduce to

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} 0 & \tau_1 g_{21} \\ \tau_2 g_{12} & 0 \end{pmatrix}, \quad \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} w - \sigma_{11} & -\sigma_{12} \\ -\sigma_{21} & w - \sigma_{22} \end{pmatrix}^{-1}, \quad (63)$$

and the corresponding equations for $\sigma_{\bar{w}\bar{w}}$ and $g_{\bar{w}\bar{w}}$ being complex conjugate of those above. This gives

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} w & -\tau_1 g_{21} \\ -\tau_2 g_{12} & w \end{pmatrix}^{-1}, \quad (64)$$

which has two solutions: one with $g_{11} = \frac{1}{w}$ and the other one with $g_{11} = w/\sqrt{\tau_1 \tau_2}$. We take the first one because it has the correct asymptotic behavior for large w . For this solution we have $g_{22} = \frac{1}{w}$ and $g_{12} = g_{21} = 0$. The holomorphic solution has to be sewed with the non-holomorphic one so that at the boundary $g_{12} = g_{21} = 0$. If we assume that these elements vanish also inside the non-holomorphic region (and correspondingly $g_{1\bar{2}} = g_{2\bar{1}} = 0$), then the equation (61) reduces to

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{1\bar{1}} & \sigma_{1\bar{2}} \\ \sigma_{21} & \sigma_{22} & \sigma_{2\bar{1}} & \sigma_{2\bar{2}} \\ \sigma_{\bar{1}1} & \sigma_{\bar{1}2} & \sigma_{\bar{1}\bar{1}} & \sigma_{\bar{1}\bar{2}} \\ \sigma_{\bar{2}1} & \sigma_{\bar{2}2} & \sigma_{\bar{2}\bar{1}} & \sigma_{\bar{2}\bar{2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & g_{2\bar{2}} & 0 \\ 0 & 0 & 0 & g_{1\bar{1}} \\ g_{2\bar{2}} & 0 & 0 & 0 \\ 0 & g_{1\bar{1}} & 0 & 0 \end{pmatrix}, \quad (65)$$

with vanishing diagonal blocks. This equation is identical to the equation with $\tau_1 = \tau_2 = 0$ and was discussed in the previous section. As we know it gives Girko-Ginibre distribution for the matrix Y and hence we obtain (5) for $X = X_1 X_2$.

One can repeat the whole reasoning for a product of more than two matrices. One finds again that the solution $1/w$ valid outside the non-holomorphic region corresponds to vanishing blocks $g_{\mu\nu} = g_{\bar{\mu}\bar{\nu}} = 0$ for $\mu \neq \nu$ and that it can be sewed with the non-holomorphic solution for which the blocks also vanish. This gives $\sigma_{ww} = \sigma_{\bar{w}\bar{w}}$ and one obtains exactly the same equations as for $\tau_1 = \dots = \tau_M = 0$. Therefore, for $M > 2$ the eigenvalue distribution of Y is also given by the Girko-Ginibre law. This result is universal: the spectrum of X is given by Eq. (5) independently of whether we multiply two Hermitian matrices, or Hermitian by generic complex, or Hermitian by anti-Hermitian etc. The limiting spectrum is always the same and differs only by finite-size effects.

One can also extend this result to purely real matrices generated from the ensemble with a measure [4]

$$d\mu(X) \sim \exp \left\{ -\frac{N}{2} \frac{1}{1-\tau^2} (\text{Tr} X X^T - \tau \text{Tr} X X) \right\} \prod_{ij} dX_{ij}. \quad (66)$$

The case $\tau = 1$ corresponds to symmetric real matrices, $\tau = -1$ to antisymmetric ones, and $\tau = 0$ to isotropic real matrices. The diagrammatic equations in the limit $N \rightarrow \infty$ are exactly the same as before, because the propagators have the same structure.

V. PROJECTION OF THE SPECTRUM OF A COMMUTATOR OF GUE MATRICES

In this section we show that the conjecture made in [14] is not true. Let us consider a matrix $X = X_1 X_2$ which is a product of two Hermitian GUE matrices X_1, X_2 . According to the formula (5), the eigenvalue density of X is $\rho_X(z, \bar{z}) = \frac{1}{2\pi|z|}$ for $|z| < 1$ and zero otherwise. The projection of this function on the real (or imaginary) axis gives

$$\rho_*(x) = \frac{1}{\pi} \ln \frac{1 + \sqrt{1 - x^2}}{|x|}, \quad (67)$$

for $-1 \leq x \leq 1$. According to [14], this result should be equal to the eigenvalue density $\rho_+(x)$ of $(X_1 X_2 + X_2^\dagger X_1^\dagger)/\sqrt{8}$ or $\rho_-(x)$ of $i(X_1 X_2 - X_2^\dagger X_1^\dagger)/\sqrt{8}$. Up to a scaling factor $\sqrt{8}$, these spectral densities are equal to the spectra of the anticommutator $\{X_1, X_2\}$ or the commutator $i[X_1, X_2]$, because $X_1 = X_1^\dagger, X_2 = X_2^\dagger$. Moreover, $\rho_-(x) = \rho_+(x)$ as follows from the observation that in the limit $N \rightarrow \infty$ all the moments of the commutator and the anticommutator are the same: $\text{Tr} \langle [X_1, X_2]^k \rangle = \text{Tr} \langle \{X_1, X_2\}^k \rangle$ for all $k = 1, 2, \dots$.

We calculate now the eigenvalue density $\rho_+(x)$ of the rescaled anticommutator $\{X_1, X_2\}/\sqrt{8}$. We define two matrices $A = (X_1 + X_2)/\sqrt{2}$ and $B = (X_1 - X_2)/\sqrt{2}$ which are also mutually independent Hermitian matrices with a factorized probability measure

$$d\mu(A, B) \propto e^{-N/2 \text{Tr} A^2} e^{-N/2 \text{Tr} B^2} D A D B. \quad (68)$$

We have $\{X_1, X_2\} = A^2 - B^2$. One can use the technique of free random variables [28] to calculate the eigenvalue density of $A^2 - B^2$ since in the limit $N \rightarrow \infty$ the matrices A^2 and B^2 represent free random variables. The addition law for a sum of free variables is expressed in terms of an R -transform or equivalently in terms of a Blue's function $B(z)$ which is a functional inverse of the Green's function $G(B(z)) = z$ and takes a simple form $B_{a+b}(z) = B_a(z) + B_b(z) - z^{-1}$, where a and b are free random variables. In our case $a = A^2, b = -B^2$. The Green's function G_a of A^2 is a special case of the Green's function for Wishart distribution, while G_b for $-B^2$ corresponds to a reflected Wishart spectrum $\lambda \rightarrow -\lambda$, and hence

$$G_a(z) = \frac{1 - \sqrt{1 - 4/z}}{2}, \quad G_b(z) = -G_a(-z) = \frac{-1 + \sqrt{1 + 4/z}}{2}. \quad (69)$$

The Blue functions for both cases read

$$B_a(z) = \frac{1}{z(1-z)}, \quad B_b(z) = \frac{1}{z(1+z)}, \quad (70)$$

and thus

$$B_{a+b}(z) = B_a + B_b - \frac{1}{z} = \frac{1 + z^2}{z(1 - z^2)}. \quad (71)$$

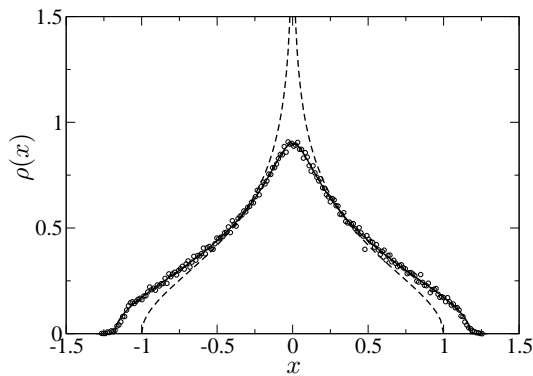


FIG. 5: Comparison between $\rho_+(x)$ from Eq. (74) (solid line), $\rho_*(x)$ from Eq. (67) (dashed line), and numerical simulations (circles) for $N = 100$ (1000 matrices were generated).

This equation has to be inverted for $G_{a+b}(z)$ which is the Green's function for the anticommutator:

$$z = \frac{1 + G_{a+b}(z)^2}{G_{a+b}(z)(1 - G_{a+b}(z)^2)}, \quad (72)$$

which leads to a cubic equation for $G_{a+b}(z)$. The solution which has the correct behavior $G_{a+b}(z) \rightarrow 1/z$ for large z reads

$$G_{a+b}(z) = \frac{1 + 3z^2 + (-1 - 18z^2 + 3\sqrt{3}\sqrt{z^2 + 11z^4 - z^6})^{2/3}}{3z(-1 - 18z^2 + 3\sqrt{3}\sqrt{z^2 + 11z^4 - z^6})^{1/3}}. \quad (73)$$

Taking into account the scaling factor $\sqrt{8}$ we finally arrive at

$$\rho_+(x) = -\frac{\sqrt{8}}{\pi} \text{Im} G_{a+b}(x\sqrt{8} + i0+) = \frac{\sqrt{3}}{6\pi} \frac{1 + 24x^2 - (1 + 144x^2 - 6\sqrt{6}\sqrt{x^2 + 88x^4 - 64x^6})^{2/3}}{|x|(1 + 144x^2 - 6\sqrt{6}\sqrt{x^2 + 88x^4 - 64x^6})^{1/3}}. \quad (74)$$

This is different from $\rho_*(x)$ from Eq. (67). In Fig. 5 we compare both spectral densities and show also results of numerical simulations which perfectly agree with (74). This falsifies the conjecture that if the spectrum of a non-Hermitian matrix is rotationally symmetric, it can be found by solving the symmetrized or antisymmetrized Hermitian problem.

VI. CONCLUSIONS

The main result of this paper is that the eigenvalue density of a product of large, centered (with zero mean) Gaussian matrices assumes a very universal form (5) with a single scaling parameter σ representing the radius of a circular support in the complex plane and related to the amplitude of fluctuations of matrix entries. The matrices in the product do not have to be identical and each of them may belong to a different elliptic ensemble.

Taking into account the universality of the Wigner's semicircle law or the Girko-Ginibre distribution for matrices having their entries drawn from independent distributions, it is tempting to conjecture that our result will also hold in this setting. Namely, we suppose that the same asymptotic result holds for products of Wigner matrices having independent elements drawn from any centered distribution which fulfills Pastur-Lindeberg's condition [3]. To assess the validity of this conjecture we performed numerical simulations, assuming various distributions of elements of the matrices. The only requirement was that the variance of the distribution was equal to $1/N$. We did not observe any deviations from (5) for short-tailed distributions. In Fig. 6 we show an example for a uniform distribution with zero mean and variance $1/N$.

As far as future projects are concerned, it would be interesting to generalize the discussion to the Gaussian symplectic ensemble [21] and to study microscopic properties of eigenvalues of the product of various types of Gaussian matrices from different invariant ensembles [20–22]. It would also be interesting to analytically derive the formula for the eigenvalue distribution of the product of M matrices of finite size N (see Fig. 2 in the middle). For the Girko-Ginibre ensemble [29] it is given by $\rho(z) \cong \text{erfc}(\sqrt{2}(|z| - 1)\sqrt{N})/(2\pi)$. We expect a qualitatively similar behavior also for the product of matrices.

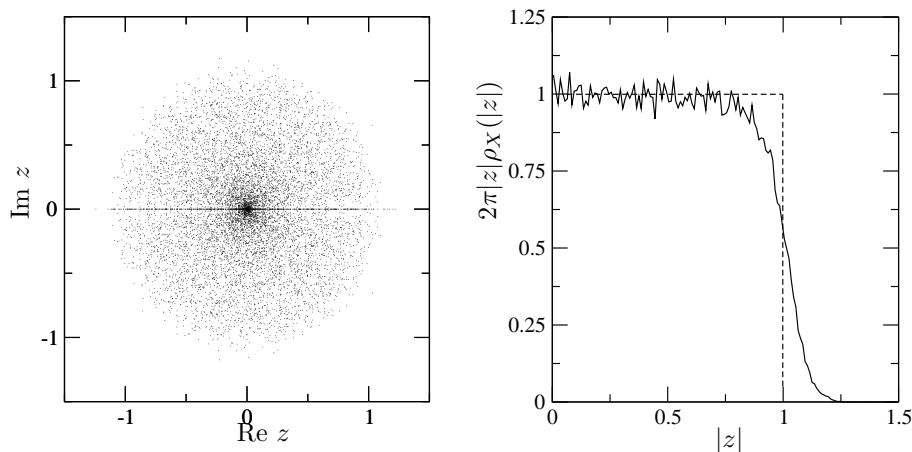


FIG. 6: Plots of numerically obtained $\rho_X(|z|)$ for X_1, X_2 being two symmetric matrices which entries (upper triangle) are taken from uniform distribution $[-\sqrt{3/N}, \sqrt{3/N}]$, for $N = 200$ and for 1000 matrices generated. Dashed line shows the theoretical distribution in the limit $N \rightarrow \infty$.

The discussion presented in this paper holds for Gaussian matrices for which the first moment has zero mean, $\langle \text{Tr} X_\mu \rangle = 0$. It would be interesting to check how it changes when $\langle \text{Tr} X_\mu \rangle \neq 0$. This could be the first step towards a generalization of Voiculescu's S -transform composition rule [30] for calculating the eigenvalue density of asymptotically large matrices representing free random variables, to the case when their product has complex eigenvalues.

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